Announcements

1) Midterm (take-home portion) should be up Wednesday

Matlab Calling Commands for Polynomial Data-Fitting

c = polyfit(x, y, d)

degree of polynomial

x-coordinates of data

y-coordinates of data

p = polyval(c', x)

variable

Conditioning

We begin to turn our attention to computational aspects.

"Conditioning" refers to sensitivity of a function to "small" changes in input data.

"Stability" will refer to essentially the same thing, but with respect to an algorithm

Example 1:

$$A = \begin{bmatrix} 1 & 2 \\ -1 & i \end{bmatrix}$$

Choose the 1-norm on $\mathbb{C}^2$.

If $x, y \in \mathbb{C}^2$,

$$\begin{aligned} & A(x+y) - Ax \\ &= \cancel{Ax} + Ay - \cancel{Ax} \\ &= Ay \end{aligned}$$

Then $(y \neq 0)$

$$\frac{\| A(x+y) - Ax \|_1}{\| y \|_1}$$

$$= \frac{\| Ay \|_1}{\| y \|_1}$$

$$\leq \frac{\| A \|_1 \cancel{\| y \|_1}}{\cancel{\| y \|_1}}$$ (by definition of $\|A\|_1$)

$$= \| A \|_1 = 3$$

This tells you that the difference between the quantity $A(x+y)$ and Ax, relative to the length of y, is never more than 3!

Recall: all norms on a finite dimensional vector space are equivalent. So in general, up to some absolute constant (dimension dependent), the choice between two norms is irrelevant.

The Set-Up

V and W are normed linear spaces.

$f: V \to W$ some function. Abusively denote both the norm on V and the norm on W by $\|\cdot\|$.

Definition: (condition numbers)

The condition number of $f: V \to W$ is the number

$$\boxed{\lim_{\delta \to 0} \sup_{\|\delta x\| < \delta} \frac{\|\delta f\|}{\|\delta x\|}}$$

where $\delta x \in V$ and

$$\delta f = f(x + \delta x) - f(x)$$

If $x \in \mathcal{V}$, the relative condition number is the quantity

$$\boxed{\kappa = \lim_{\delta \to 0} \sup_{\|\delta x\| < \delta} \frac{\left(\|\delta f\| / \|\delta x\| \right)}{\left(\|f(x)\| / \|x\| \right)}}$$

Example 2: (any linear map)

Let V and W be finite-dimensional normed linear spaces over $\mathbb{R}$ or $\mathbb{C}$.

Let $L: V \to W$ be linear.

Then L can be represented as a matrix A with

$$Ax = Lx \quad \forall\, x \in V.$$

The condition number of L is then the condition number of A, which is:

$$\lim_{\delta \to 0} \sup_{\|\delta x\| < \delta} \frac{\| A(x+\delta x) - Ax \|}{\| \delta x \|}$$

$$= \lim_{\delta \to 0} \sup_{\|\delta x\| < \delta} \frac{\| \cancel{Ax} + A\delta x - \cancel{Ax} \|}{\| \delta x \|}$$

$$= \lim_{\delta \to 0} \sup_{\|\delta x\| < \delta} \frac{\| A \delta x \|}{\| \delta x \|}$$

$$= \| A \| \quad \text{(check definition)}$$

The Jacobian and Conditioning

For the definitions of condition numbers, f need not even be continuous, let alone differentiable!

Recall: If V and W are finite dimensional spaces with norms abusively both denoted $\|\cdot\|$, f is differentiable at $x \in V$ if $\exists$ a linear map $L: V \to W$ such that

$$\boxed{\lim_{h \to 0} \frac{\|f(x+h) - f(x) - Lh\|}{\|h\|} = 0}$$

L is called the derivative of f

at x, and as such,

may be represented

as a matrix

$A: V \to W$.

Suppose $V = \mathbb{C}^n$, $W = \mathbb{C}^m$.

Then $A \in \mathbb{C}^{m \times n}$ and

$$\boxed{A_{i,j} = \frac{\partial f_i}{\partial x_j}(x)}$$

for all $1 \leq i \leq m$

and $1 \leq j \leq n$.

The idea: if

f is differentiable,

then as $\|\delta x\|$ becomes small,

$$\frac{\|\delta f\|}{\|\delta x\|} = \frac{\|f(x+\delta x) - f(x)\|}{\|\delta x\|}$$

$$\approx \frac{\|A\,\delta x\|}{\|\delta x\|}$$

where A is the matrix of partial derivatives.

Then the condition number of f is

$$\lim_{\delta \to 0} \sup_{\|\delta x\| < \delta} \frac{\|\delta f\|}{\|\delta x\|}$$

$$= \lim_{\delta \to 0} \sup_{\|\delta x\| < \delta} \frac{\|A\,\delta x\|}{\|\delta x\|}$$

$$= \|A\|$$

We will call A instead by the letter J, for Jacobian

Then for differentiable f,

$$K = \lim_{\delta \to 0} \sup_{\|\delta x\| < \delta} \frac{\left(\frac{\|\delta f\|}{\|\delta x\|} \right)}{\left(\frac{\|f(x)\|}{\|x\|} \right)}$$

$$= \frac{\|J\| \cdot \|x\|}{\|f(x)\|}.$$

Definition: (well/ill-conditioned)

A problem given by a function $f: V \to W$ where V and W are normed linear spaces is called well-conditioned if K is "small" relative to a given tolerance.

The problem is ill-conditioned if K is not small relative to the given tolerance.